WEDGE DECOMPOSITION OF POLYHEDRAL PRODUCTS

KOUYEMON IRIYE AND DAISUKE KISHIMOTO

Abstract. We prove that certain polyhedral products, including the moment-angle complexes, for the Alexander duals of shellable and sequentially Cohen-Macaulay complexes decompose into wedges of explicitly given suspension spaces, which implies the properties of the Stanley-Reisner rings, known as Golod, of theses complexes.

1. Introduction

The *polyhedral product* $\mathcal{Z}_K(\underline{X},\underline{A})$ is a space constructed from a simplicial complex K on the index set $[m]=\{1,\dots,m\}$, possibly with ghost vertices (i.e. elements of $[m]$ which are not vertices), and a collection of pairs of spaces $(\underline{X},\underline{A})=\{(X_i,A_i)\}_{i=1}^m$. Polyhedral products for special simplicial complexes and pairs of spaces were first found in the work of Porter [P] on higher Whitehead products. Since then, they have been studied by many topologists. After a seminal work of Davis and Januszkiewicz [DJ] on quasitoric manifolds, the polyhedral products $\mathcal{Z}_K(\underline{\mathbb{C}P}^\infty,*)$ and $\mathcal{Z}_K(\underline{D}^2,\underline{S}^1)$, called the *Davis-Januszkiewicz space* and the *moment-angle complex* for K respectively, have been gathering attention in connection with toric topology, where $(\underline{\mathbb{C}P}^\infty,*)$ and $(\underline{D}^2,\underline{S}^1)$ are m copies of pairs $(\mathbb{C}P^\infty,*)$ and (D^2,S^1), respectively. In this paper, we attempt to deduce some properties of algebras concerning simplicial complexes from homotopical properties of polyhedral products.

One of the most important algebras constructed from a simplicial complex K with no ghost vertices is the *Stanley-Reisner ring* which is defined as

$$\Bbbk[K]=\Bbbk[v_1,\dots,v_m]/\mathcal{I}_K,$$

where $\Bbbk$ is a commutative ring, the degree of v_i is 2 and $\mathcal{I}_K$ is the ideal generated by monomials $v_{i_1}\cdots v_{i_k}$ for $\{i_1,\dots,i_k\}\notin K$. By definition, the cohomology of the Davis-Januszkiewicz space for K is identified with the Stanley-Reisner ring of K. Using this identification, Buchstaber and Panov [BP] proved that the cohomology of the moment-angle complex for K is identified with the Tor-algebra $\mathrm{Tor}_{\Bbbk[v_1,\dots,v_m]}(\Bbbk[K],\Bbbk)$. We aim at proving some algebraic properties of this Tor-algebra can be deduced from much stronger homotopical properties of polyhedral products. Let us recall properties of the Tor-algebra $\mathrm{Tor}_{\Bbbk[v_1,\dots,v_m]}(\Bbbk[K],\Bbbk)$. Its graded module structure

2010 *Mathematics Subject Classification.* Primary 55P15, 13F55, Secondary 52B22.
Key words and phrases. polyhedral product; Stanley-Reisner ring; Golod ring; shellable complex; sequentially Cohen-Macaulay complex.

was determined by Hochster [H] so that there is an isomorphism of graded modules

$$\mathrm{Tor}_{\Bbbk[v_1,\ldots,v_m]}(\Bbbk[K],\Bbbk)\cong\bigoplus_{I\subset[m]}H^*(K_I;\Bbbk),\tag{1.1}$$

where K_I stands for the induced subcomplex of K on the subset $I\subset[m]$. In fact, this purely algebraic result is a consequence of the decomposition of a suspension of a polyhedral product which is due to Bahri, Bendersky, Cohen and Gitler [BBCG]. To state this decomposition, we set notation. Let $|K|$ be the geometric realization of K and let ΣK be the (unreduced) suspension of K which is the join of K and the simplicial complex consisting of discrete two points p,q, where the basepoint of $|\Sigma K|$ is chosen to be the vertex p. For a collection of spaces $\{X_i\}_{i=1}^m$ and a subset $I\subset[m]$, let $\widehat{X}^I=\bigwedge_{i\in I}X_i$, where we conventionally assume $\widehat{X}^I$ is a single point if $I=\emptyset$. We also let $(C\underline{X},\underline{X})$ be a collection $\{(CX_i,X_i)\}_{i=1}^m$.

Theorem 1.1. *Let K be a simplicial complex on the index set $[m]$, possibly with ghost vertices, and let $\underline{X}=\{X_i\}_{i=1}^m$ be a collection of connected CW-complexes. There is a homotopy equivalence*

$$\Sigma\mathcal{Z}_K(C\underline{X},\underline{X})\simeq\Sigma\bigvee_{I\subset[m]}|\Sigma K|\wedge\widehat{X}^I.$$

The congruence (1.1) can be deduced by putting $X_i=S^1$ in Theorem 1.1. Let us next consider the ring structure of $\mathrm{Tor}_{\Bbbk[v_1,\ldots,v_m]}(\Bbbk[K],\Bbbk)$. The simplest case is that all products and (higher) Massey products in $\mathrm{Tor}_{\Bbbk[v_1,\ldots,v_m]}(\Bbbk[K],\Bbbk)$ are trivial. In such a case, K is called *Golod*, where the name comes from the work of Golod [G] who proved that Golodness is equivalent to a certain equality among the Poincaré series of algebras related with $\Bbbk[K]$. This special class of simplicial complexes has been extensively studied, especially in connection with shellability (cf. [B, HRW]). Shellability is a criterion for pure complexes being Cohen-Macauly, which was generalized to the non-pure case [BW] and to a wider class of simplicial complexes called sequentially Cohen-Macaulay complexes [BWW]. The definition of shellable and sequentially Cohen-Macaulay complexes will be given in the following sections. At the moment, we only note the property that shellable complexes are sequentially Cohen-Macaulay. In [BW], two special classes of shellable complexes were introduced; shifted and vertex-decomposable complexes. It was proved in [BW] that shifted complexes are vertex-decomposable. Summarizing, there are implications:

$$\text{shifted}\ \Rightarrow\ \text{vertex-decomposable}\ \Rightarrow\ \text{shellable}\ \Rightarrow\ \text{sequentially Cohen-Macaulay}\tag{1.2}$$

As for Golodness, it was shown in [HRW] that the Alexander duals of sequentially Cohen-Macaulay complexes are Golod, hence the Alexander duals of all complexes in (1.2) are also. As the Hochster's isomorphism (1.1) is deduced from a much stronger homotopical property of polyhedral products (Theorem 1.1), we might expect that Golodness of the Alexander duals of complexes in (1.2) can also be deduced from a stronger homotopical property of polyhedral products. There are some results confirming this expectation. Grbić and Theriault [GT] proved

that the moment-angle complex for a shifted complex has the homotopy type of a wedge of spheres. This result was generalized to the polyhedral product $\mathcal{Z}_K(C\underline{X},\underline{X})$ by the authors [IK], i.e. the homotopy equivalence of Theorem 1.1 desuspends if K is shifted. We here notice that the Alexander dual of a shifted complex is shifted. Recently, by a noble use of discrete Morse theory, Grujić and Welker [GW] proved if K is the Alexander dual of a vertex-decomposable complex, the polyhedral product $\mathcal{Z}_K(\underline{D}^n,\underline{S}^{n-1})$ has the homotopy type of a wedge of spheres for $n\ge 2$. Then in these cases, Golodness can be deduced from a homotopical property of polyhedral products. By the implications (1.2), our next step should be to show that Golodness of the Alexander duals of shellable and sequentially Cohen-Macaulay complexes can be deduced from polyhedral products. We will prove:

Theorem 1.2. *Let K be a simplicial complex on the index set $[m]$ with no ghost vertex and let $\underline{X}=\{X_i\}_{i=1}^m$ be a collection of connected CW-complexes. If the Alexander dual of K is shellable, there is a homotopy equivalence*

$$\mathcal{Z}_K(C\underline{X},\underline{X})\simeq\bigvee_{I\subset[m]}|\Sigma K_I|\wedge\widehat{X}^I.$$

Theorem 1.3. *Let K and $\underline{X}$ be as in Theorem 1.2. If the Alexander dual of K is sequentially Cohen-Macaulay and each X_i is finite, there is a homotopy equivalence*

$$\mathcal{Z}_K(C\underline{X},\underline{X})\simeq\bigvee_{I\subset[m]}|\Sigma K_I|\wedge\widehat{X}^I.$$

Corollary 1.4. *If the Alexander dual of K is sequentially Cohen-Macaulay (including shellable), the moment-angle complex for K has the homotopy type of a wedge of spheres.*

Remark 1.5. For sequentially Cohen-Macaulay complexes, we will be able only to prove the p-local version of Theorem 1.2. In deducing the integral decomposition from this, we need the finiteness assumption in Theorem 1.3.

Corollary 1.4 readily implies the above mentioned result of Herzog, Reiner and Welker [HRW].

Corollary 1.6. *The Alexander duals of sequentially Cohen-Macaulay complexes are Golod.*

The paper is organized as follows. In section 2, we review the result of Bahri, Bendersky, Cohen and Gitler [BBCG] on a wedge decomposition of polyhedral products and consider its naturality which will be needed below. In section 3, we introduce a new class of simplicial complexes called extractible by a recursive condition on deletions of vertices, which abstracts a necessary condition to yield a wedge decomposition of polyhedral products for the Alexander duals of shellable and sequentially Cohen-Macaulay complexes. Then we prove a wedge decomposition of polyhedral products for extractible complexes. In section 4, we show that the Alexander duals of shellable complexes are extractible by analyzing the Alexander dual of subcomplexes of shellable complexes obtained by removing homotopically redundant facets, and then we prove Theorem 1.2. In section 5, we prove extractibility of the Alexander duals of

sequentially Cohen-Macaulay complexes by the homological analogue of the method employed in section 4. As in Remark 1.5, we can only show extractibility of the Alexander duals of sequentially Cohen-Macaulay complexes over the local ring $\mathbb{Z}_{(p)}$, so we integrate the p-local information to get Theorem 1.3 by using the result of McGibbon [M] on the localization genus (or the Mislin genus).

2. Review of the result of Bahri, Bendersky, Cohen and Gitler

Hereafter, we assume that spaces have non-degenerate basepoints and maps between spaces preserve basepoints. We also assume that the empty set is a simplex of a simplicial complex.

Let us first define polyhedral products. Let K be a simplicial complex on the index set $[m]$, possibly with ghost vertices, and let $(\underline{X},\underline{A})$ be a collection of pairs of spaces $\{(X_i,A_i)\}_{i=1}^m$. For a simplex $\sigma\in K$, put

$$(\underline{X},\underline{A})^\sigma = Y_1\times\cdots\times Y_m,\quad \text{where}\quad Y_i=\begin{cases}X_i & i\in\sigma\\ A_i & i\notin\sigma.\end{cases} \tag{2.1}$$

The polyhedral product $\mathcal{Z}_K(\underline{X},\underline{A})$ is defined as

$$\mathcal{Z}_K(\underline{X},\underline{A})=\bigcup_{\sigma\in K}(\underline{X},\underline{A})^\sigma,$$

where the unions taken in $\prod_{i=1}^m X_i$. Let us consider natural maps between polyhedral products. For a subcomplex $L\subset K$ and a collection of maps between pairs of spaces $\underline{f}=\{f_i:(X_i,A_i)\to(Y_i,B_i)\}_{i=1}^m$ with $(\underline{Y},\underline{B})=\{(Y_i,B_i)\}_{i=1}^m$, there are induced maps

$$\mathcal{Z}_L(\underline{X},\underline{A})\to\mathcal{Z}_K(\underline{X},\underline{A})\quad\text{and}\quad \underline{f}:\mathcal{Z}_K(\underline{X},\underline{A})\to\mathcal{Z}_K(\underline{Y},\underline{B}).$$

For a subset $I\subset[m]$, let $(\underline{X}_I,\underline{A}_I)$ be a subcollection $\{(X_i,A_i)\}_{i\in I}$ of $(\underline{X},\underline{A})$. By definition, the projection $\prod_{i=1}^m X_i\to\prod_{i\in I}X_i$ restricts to a map

$$\pi_I:\mathcal{Z}_K(\underline{X},\underline{A})\to\mathcal{Z}_{K_I}(\underline{X}_I,\underline{A}_I).$$

Replacing the direct product in (2.1) with the smash product, we can also define the smash product analogue of $\mathcal{Z}_K(\underline{X},\underline{A})$ which we denote by $\widehat{\mathcal{Z}}_K(\underline{X},\underline{A})$. For a subcomplex $L\subset K$ and a map $\underline{f}:(\underline{X},\underline{A})\to(\underline{Y},\underline{B})$, there are also induced maps

$$\widehat{\mathcal{Z}}_L(\underline{X},\underline{A})\to\widehat{\mathcal{Z}}_K(\underline{X},\underline{A})\quad\text{and}\quad \underline{f}:\widehat{\mathcal{Z}}_K(\underline{X},\underline{A})\to\widehat{\mathcal{Z}}_K(\underline{Y},\underline{B}).$$

The pinch map $\prod_{i=1}^m X_i\to\bigwedge_{i=1}^m X_i$ restricts to a map

$$\rho_I:\mathcal{Z}_K(\underline{X},\underline{A})\to\widehat{\mathcal{Z}}_K(\underline{X},\underline{A}).$$

Let $\nabla:\Sigma X\to\Sigma X\vee\Sigma X$ be the suspension comultiplication and let ∇_n be the composite

$$\Sigma X\xrightarrow{\nabla}\Sigma X\vee\Sigma X\xrightarrow{1\vee\nabla}\cdots\xrightarrow{1\vee\cdots\vee 1\vee\nabla}\underbrace{\Sigma X\vee\cdots\vee\Sigma X}_{n}$$

for $n \geq 2$. Let $I_1 < \cdots < I_{2^m} \subset [m]$ be the lexicographic order on the power set of $[m]$. We now define the map

$$\hat{\epsilon}_K = (\Sigma(\rho_{I_1} \circ \pi_{I_1}) \vee \cdots \vee \Sigma(\rho_{I_{2^m}} \circ \pi_{I_{2^m}})) \circ \nabla_{2^m} : \Sigma\mathcal{Z}_K(\underline{X}, \underline{A}) \to \Sigma \bigvee_{I \subset [m]} \widehat{\mathcal{Z}}_{K_I}(\underline{X}_I, \underline{A}_I).$$

Generalizing the standard decomposition $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$, i.e. the composite

$$\Sigma(X \times Y) \xrightarrow{\nabla_3} \bigvee^{3} \Sigma(X \times Y) \xrightarrow{\mathrm{proj}} \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y), \tag{2.2}$$

Bahri, Bendersky, Cohen and Gitler [BBCG] proved:

Theorem 2.1. *The map $\hat{\epsilon}_K$ is a homotopy equivalence if each (X_i, A_i) is a connected CW-pair.*

Remark 2.2. Notice that $\hat{\epsilon}_K$ is defined by using the lexicographic order on the power set of $[m]$. If we apply an alternative order on the power set of $[m]$, we get an alternative homotopy equivalence. However we easily see that these homotopy equivalences become homotopic after a suspension by the cocommutativity of a double suspension.

From now on, we fix a collection of spaces $\underline{X} = \{X_i\}_{i=1}^m$ and specialize polyhedral products and related spaces to the collection $(C\underline{X}, \underline{X})$. Then it is useful to put for $I \subset [m]$,

$$\mathcal{Z}_K^I = \mathcal{Z}_{K_I}(C\underline{X}_I, \underline{X}_I), \quad \widehat{\mathcal{Z}}_K^I = \widehat{\mathcal{Z}}_{K_I}(C\underline{X}_I, \underline{X}_I) \quad \text{and} \quad \mathcal{W}_K^I = \bigvee_{J \subset I} |\Sigma K_J| \wedge \widehat{X}^J.$$

Using (pointed) homotopy colimits, it is proved in [ZZ] that there is a homotopy equivalence

$$\varpi_I : \widehat{\mathcal{Z}}_K^I \xrightarrow{\simeq} |\Sigma K_I| \wedge \widehat{X}^I.$$

Putting

$$\bar{\epsilon}_K = ((\bigvee_{I \subset [m]} \Sigma\varpi_I) \circ \hat{\epsilon}_K)^{-1} : \Sigma\mathcal{W}_K^{[m]} \to \Sigma\mathcal{Z}_K^{[m]}, \tag{2.3}$$

we get a homotopy equivalence of Theorem 1.1. Let us consider the case that $\bar{\epsilon}_K$ desuspends. If $\mathcal{Z}_K(\underline{X}, \underline{A})$ is a co-H-space, the map ∇_{2^m} in the definition of $\hat{\epsilon}_K$ desuspends, i.e. there is a map $\mathcal{Z}_K(\underline{X}, \underline{A}) \to \bigvee^{2^m} \mathcal{Z}_K(\underline{X}, \underline{A})$ whose suspension is homotopic to ∇_{2^m}. Then, in particular, we have:

Proposition 2.3. *If $\mathcal{Z}_K^{[m]}$ is a co-H-space, $\bar{\epsilon}_K$ desuspends.*

Let us consider the naturality of $\bar{\epsilon}_K$. We start with the map $\hat{\epsilon}_K$. By definition, $\hat{\epsilon}_K$ has the naturality such that for a subcomplex $L \subset K$ and a map $\underline{f} : (\underline{X}, \underline{A}) \to (\underline{Y}, \underline{B})$, there are homotopy commutative squares

(2.4)

$$\begin{array}{ccc} \Sigma\mathcal{Z}_L(\underline{X}, \underline{A}) & \xrightarrow{\hat{\epsilon}_L} & \Sigma\bigvee_{I \subset [m]} \widehat{\mathcal{Z}}_{L_I}(\underline{X}_I, \underline{A}_I) \\ \downarrow{\scriptstyle \mathrm{incl}} & & \downarrow{\scriptstyle \mathrm{incl}} \\ \Sigma\mathcal{Z}_K(\underline{X}, \underline{A}) & \xrightarrow{\hat{\epsilon}_K} & \Sigma\bigvee_{I \subset [m]} \widehat{\mathcal{Z}}_{K_I}(\underline{X}_I, \underline{A}_I) \end{array} \quad \text{and} \quad \begin{array}{ccc} \Sigma\mathcal{Z}_L(\underline{X}, \underline{A}) & \xrightarrow{\hat{\epsilon}_L} & \Sigma\bigvee_{I \subset [m]} \widehat{\mathcal{Z}}_{L_I}(\underline{X}_I, \underline{A}_I) \\ \downarrow{\scriptstyle \underline{f}} & & \downarrow{\scriptstyle \bigvee_{I \subset [m]} \underline{f}_I} \\ \Sigma\mathcal{Z}_K(\underline{Y}, \underline{B}) & \xrightarrow{\hat{\epsilon}_K} & \Sigma\bigvee_{I \subset [m]} \widehat{\mathcal{Z}}_{K_I}(\underline{Y}_I, \underline{B}_I) \end{array}$$

where $\underline{f}_I$ is a subcollection of $\underline{f}$ corresponding to I. If v is a ghost vertex of K, by the same reason as Remark 2.2, the following square becomes homotopy commutative after a suspension.

$$\begin{tikzcd}
\Sigma\mathcal{Z}_K(\underline{X},\underline{A}) \arrow[r, equal] \arrow[dd, "\hat{\epsilon}_K"] & \Sigma(\mathcal{Z}_K(\underline{X}_{[m]\backslash v},\underline{A}_{[m]\backslash v})\times A_v) \arrow[d, "\hat{\delta}"] \\
 & \Sigma\mathcal{Z}_K(\underline{X}_{[m]\backslash v},\underline{A}_{[m]\backslash v})\vee\Sigma A_v\vee\Sigma(\mathcal{Z}_K(\underline{X}_{[m]\backslash v},\underline{A}_{[m]\backslash v})\wedge A_v) \arrow[d, "\hat{\epsilon}_K\vee 1\vee(\hat{\epsilon}_K\wedge 1)"] \\
\Sigma\bigvee_{I\subset[m]}\widehat{\mathcal{Z}}_{K_I}(\underline{X}_I,\underline{A}_I) \arrow[r, equal] & \Sigma\widehat{\mathcal{Z}}_K(\underline{X}_{[m]\backslash v},\underline{A}_{[m]\backslash v})\vee\Sigma A_v\vee\Sigma(\widehat{\mathcal{Z}}_K(\underline{X}_{[m]\backslash v},\underline{A}_{[m]\backslash v})\wedge A_v),
\end{tikzcd} \tag{2.5}$$

where $\hat{\delta}$ is the composite (2.2). We next consider the naturality of ϖ_I. By definition, ϖ_I has the naturality analogous to (2.4). Moreover, if $v\in[m]$ is a ghost vertex of K, there is a homotopy commutative diagram

$$\begin{tikzcd}
\widehat{\mathcal{Z}}_K^{[m]\backslash v}\wedge X_v \arrow[r, equal] \arrow[d, "\varpi_{[m]\backslash v}\wedge 1"] & \widehat{\mathcal{Z}}_K^{[m]} \arrow[d, "\varpi_{[m]}"] \\
(|\Sigma K|\wedge\widehat{X}^{[m]\backslash v})\wedge X_v \arrow[r, equal] & |\Sigma K|\wedge\widehat{X}^{[m]}.
\end{tikzcd} \tag{2.6}$$

We here record the naturality of $\bar{\epsilon}_K$ which will be used below. Let $\delta:\Sigma X\rtimes Y\to\Sigma X\vee(\Sigma X\wedge Y)$ be a homotopy equivalence defined as the composite

$$\Sigma X\rtimes Y\xrightarrow{\nabla}(\Sigma X\rtimes Y)\vee(\Sigma X\rtimes Y)\xrightarrow{\mathrm{proj}}\Sigma X\vee(\Sigma X\wedge Y), \tag{2.7}$$

where ∇ is the suspension comultiplication.

Proposition 2.4. *For a subcomplex $L\subset K$ and a subset $I\subset[m]$, there are homotopy commutative diagrams*

$$\begin{tikzcd}
\mathcal{W}_L^{[m]} \arrow[r, "\mathrm{incl}"] \arrow[d, "\bar{\epsilon}_L"] & \mathcal{W}_K^{[m]} \arrow[d, "\bar{\epsilon}_K"] \\
\mathcal{Z}_L^{[m]} \arrow[r, "\mathrm{incl}"] & \mathcal{Z}_K^{[m]}
\end{tikzcd} \quad\text{and}\quad \begin{tikzcd}
\mathcal{W}_{K_I}^{I} \arrow[r, "\mathrm{incl}"] \arrow[d, "\bar{\epsilon}_{K_I}"] & \mathcal{W}_K^{[m]} \arrow[d, "\bar{\epsilon}_K"] \\
\mathcal{Z}_{k_I}^{I} \arrow[r, "\mathrm{incl}"] & \mathcal{Z}_K^{[m]}.
\end{tikzcd}$$

Moreover, if $v\in[m]$ is a ghost vertex of K, the following diagram becomes homotopy commutative after a suspension.

$$\begin{tikzcd}
\Sigma\mathcal{W}_K^{[m]} \arrow[r, "\mathrm{proj}"] \arrow[d, "\bar{\epsilon}_K"] & \Sigma\mathcal{W}_K^{[m]\backslash v}\vee\Sigma(\mathcal{W}_K^{[m]\backslash v}\wedge X_v) \arrow[r, "\delta^{-1}"] & \Sigma\mathcal{W}_K^{[m]\backslash v}\rtimes X_v \arrow[d, "\bar{\epsilon}_K\rtimes 1"] \\
\Sigma\mathcal{Z}_K^{[m]} \arrow[r, equal] & \Sigma(\mathcal{Z}_K^{[m]\backslash v}\times X_v) \arrow[r, "\mathrm{proj}"] & \Sigma\mathcal{Z}_K^{[m]\backslash v}\rtimes X_v
\end{tikzcd}$$

Proof. The first two squares follow from the combination of (2.4) and its analogue for ϖ_K. Consider the following diagram.

$$\begin{tikzcd}
\Sigma\mathcal{W}_K^{[m]} \arrow[r, "\mathrm{proj}"] \arrow[d, "\bigvee_{I\subset[m]}\varpi_I^{-1}"] & \Sigma\mathcal{W}_K^{[m]\setminus v}\vee\Sigma(\mathcal{W}_K^{[m]\setminus v}\wedge X_v) \arrow[r, "\delta^{-1}"] & \Sigma\mathcal{W}_K^{[m]\setminus v}\rtimes X_v \arrow[d, "(\bigvee_{I\subset[m]\setminus v}\varpi_I^{-1})\rtimes 1"] \\
\Sigma\bigvee_{I\subset[m]}\widehat{\mathcal{Z}}_K^I \arrow[r, "\mathrm{proj}"] \arrow[d, "\hat{\epsilon}_K^{-1}"] & \Sigma\bigvee_{\substack{I\subset[m]\\ I\neq v}}\widehat{\mathcal{Z}}_K^I \arrow[r, "\delta^{-1}"] & \Sigma\bigvee_{I\subset[m]\setminus v}\widehat{\mathcal{Z}}_K^I\rtimes X_v \arrow[d, "\hat{\epsilon}_K^{-1}\rtimes 1"] \\
\Sigma\mathcal{Z}_K^{[m]} \arrow[r, equal] & \Sigma(\mathcal{Z}_K^{[m]\setminus v}\times X_v) \arrow[r, "\mathrm{proj}"] & \Sigma\mathcal{Z}_K^{[m]\setminus v}\rtimes X_v
\end{tikzcd}$$

The upper diagram is homotopy commutative by (2.6) and the lower square becomes homotopy commutative after a suspension by (2.5). Therefore by the definition of $\bar{\epsilon}_K$, we obtain the third naturality. □

We close this section by evaluating the connectivity of $\mathcal{Z}_K^{[m]}$.

Proposition 2.5. *If K has no ghost vertex and each X_i is path-connected, $\mathcal{Z}_K^{[m]}$ is simply connected.*

Proof. For a subset $I\subset[m]$, we write the simplicial complex on the index $[m]$ consisting of discrete vertices in I ambiguously by I. For a simplex $\sigma\in K$, we put

$$D(\sigma)=\mathcal{Z}_{[m]}^{[m]}\cup(C\underline{X},\underline{X})^{\sigma}.$$

As in [P] (cf. [IK]), $\mathcal{Z}_{[m]}^{[m]}$ is simply connected, hence so is $D(\sigma)$ by the van Kampen theorem. By definition, we have $\mathcal{Z}_K^{[m]}=\bigcup_F D(F)$, where F ranges over all facets of K. We prove the proposition by induction on the number of facets of K. If K has only one facet, K is a simplex, implying that $\mathcal{Z}_K^{[m]}$ is contractible hence simply connected. If $K=[m]$, $\mathcal{Z}_K^{[m]}$ is simply connected as above. Then we may assume there is a facet F with $\dim F\geq 1$, that is, $K\setminus F$ has no ghost vertex. By the induction hypothesis, $\mathcal{Z}_{K\setminus F}^{[m]}$ is simply connected. Thus since $\mathcal{Z}_{K\setminus F}^{[m]}\cap D(\sigma)$ is path-connected and $D(F)$ is simply connected, the result follows from the van Kampen theorem. □

Corollary 2.6. *If $\mathcal{Z}_K^{[m]}$ is a co-H-space and each X_i is a connected CW-complex, there is a homotopy equivalence*

$$\mathcal{Z}_K^{[m]}\simeq\mathcal{W}_K^{[m]}.$$

Proof. By Proposition 1.1, the map of Proposition 2.3 induces an isomorphism in homology. Thus the proof is completed by Proposition 2.5 and the J.H.C. Whitehead theorem. □

3. Extractible complexes

We first set notation for simplicial complexes. Let K be a simplicial complex on the index set $[m]$, possibly with ghost vertices. The link, the star and the deletion of a simplex $\sigma \in K$ is defined respectively as

$$\begin{aligned}
\mathrm{lk}_K(\sigma) &= \{\tau \in K \,|\, \sigma \cup \tau \in K,\ \sigma \cap \tau = \emptyset\},\\
\mathrm{st}_K(\sigma) &= \{\tau \in K \,|\, \sigma \cup \tau \in K\},\\
\mathrm{dl}_K(\sigma) &= \{\tau \in K \,|\, \sigma \not\subset \tau\}.
\end{aligned}$$

The Alexander dual of K is defined as

$$K^\vee = \{\sigma \subset [m], |\, [m] \setminus \sigma \notin K\}.$$

Since Alexander duals depend on index sets, we must be careful for them.

We now introduce extractible complexes.

Definition 3.1. A simplicial complex K with no ghost vertex is called *extractible* over $\Bbbk$ if

(1) K consists of a single simplex $\emptyset$ or $\mathrm{dl}_K(v)$ is a simplex for some vertex v, or

(2) $\mathrm{dl}_K(v)$ is extractible over $\Bbbk$ for any vertex v and there is a map $|\Sigma K| \to \bigvee_{v\in[m]} |\Sigma \mathrm{dl}_K(v)|$ satisfying that the composite with the wedge of inclusions
$$|\Sigma K| \to \bigvee_{v\in[m]} |\Sigma \mathrm{dl}_K(v)| \to |\Sigma K|$$
induces the identity map in homology with $\Bbbk$ coefficient.

If K is extractible over the ring of integers $\mathbb{Z}$, we simply say K is extractible. We prove a wedge decomposition of polyhedral products for extractible complexes.

Theorem 3.2. *Let K be an extractible complex over $\Bbbk$ on the index set $[m]$. There is a map*
$$\epsilon_K : \mathcal{W}_K^{[m]} \to \mathcal{Z}_K^{[m]}$$
inducing the same map as $\bar{\epsilon}_K$ of (2.3) in homology with $\Bbbk$ coefficient.

Proof. Induct on m. If $m = 1$, both $\mathcal{W}_K^{[m]}$ and $\mathcal{Z}_K^{[m]}$ are contractible, hence the constant map is the desired ϵ_K. Suppose we have proved the case $m-1$ and then consider the case m. Suppose $\mathrm{dl}_K(v)$ is a simplex for some vertex v. Consider the pushout

$$\begin{tikzcd}
\mathcal{Z}^{[m]}_{\mathrm{lk}_K(v)} \arrow[r] \arrow[d] & \mathcal{Z}^{[m]}_{\mathrm{st}_K(v)} \arrow[d] \\
\mathcal{Z}^{[m]}_{\mathrm{dl}_K(v)} \arrow[r] & \mathcal{Z}^{[m]}_{K}
\end{tikzcd} \tag{3.1}$$

induced from the corresponding pushout of simplicial complexes, where arrows are inclusions. Note that

$$\mathcal{Z}^{[m]}_{\mathrm{lk}_K(v)} = \mathcal{Z}^{[m]\setminus v}_{\mathrm{lk}_K(v)} \times X_v, \quad \mathcal{Z}^{[m]}_{\mathrm{st}_K(v)} = \mathcal{Z}^{[m]\setminus v}_{\mathrm{lk}_K(v)} \times CX_v \quad \text{and} \quad \mathcal{Z}^{[m]}_{\mathrm{dl}_K(v)} = \mathcal{Z}^{[m]\setminus v}_{\mathrm{dl}_K(v)} \times X_v.$$

Include the pushout

$$\begin{tikzcd} X_v \arrow[r] \arrow[d, equal] & CX_v \arrow[d, equal] \\ X_v \arrow[r] & CX_v \end{tikzcd}$$

into (3.1) and take the cofiber of each corner. Then we get a pushout

$$\begin{tikzcd} \mathcal{Z}^{[m]\setminus v}_{\mathrm{lk}_K(v)} \rtimes X_v \arrow[r] \arrow[d] & \mathcal{Z}^{[m]\setminus v}_{\mathrm{lk}_K(v)} \rtimes CX_v \arrow[d] \\ \mathcal{Z}^{[m]\setminus v}_{\mathrm{dl}_K(v)} \rtimes X_v \arrow[r] & \mathcal{Z}^{[m]}_K/CX_v \end{tikzcd} \tag{3.2}$$

Since $\mathrm{dl}_K(v)$ is a simplex by assumption, $\mathcal{Z}^{[m]\setminus v}_{\mathrm{dl}_K(v)} = \prod_{i\in[m]\setminus v} CX_v$ which is contractible, hence so is $\mathcal{Z}^{[m]\setminus v}_{\mathrm{dl}_K(v)} \rtimes X_v$. Then it follows from (3.2) that there is a homotopy equivalence $\mathcal{Z}^{[m]}_K/CX_v \simeq \Sigma(\mathcal{Z}^{[m]\setminus v}_{\mathrm{lk}_K(v)} \wedge X_v)$, so $\mathcal{Z}^{[m]}_K$ has the homotopy type of a suspension. Thus by Corollary 2.6, we obtain the desired result.

Suppose next that $\mathrm{dl}_K(v)$ is extractible over $\Bbbk$ for any vertex v and there is a map $s : |\Sigma K| \to \bigvee_{v\in[m]} |\Sigma \mathrm{dl}_K(v)|$ satisfying that the composite with the wedge of inclusions

$$|\Sigma K| \xrightarrow{s} \bigvee_{v\in[m]} |\Sigma \mathrm{dl}_K(v)| \to |\Sigma K|$$

induces the identity map in homology with $\Bbbk$ coefficient. By the induction hypothesis, there is a map $\epsilon_{\mathrm{dl}_K(v)} : \mathcal{W}^{[m]\setminus v}_{\mathrm{dl}_K(v)} \to \mathcal{Z}^{[m]\setminus v}_{\mathrm{dl}_K(v)}$ with the desired property for any $v \in [m]$. Then by Proposition 2.4, the composite

$$\bigvee_{I\subsetneq[m]} |\Sigma K_I| \wedge \widehat{X}^I \xrightarrow{\mathrm{incl}} \bigvee_{v\in[m]} \mathcal{W}^{[m]\setminus v}_{\mathrm{dl}_K(v)} \xrightarrow{\bigvee_{v\in[m]}\epsilon_{\mathrm{dl}_K(v)}} \bigvee_{v\in[m]} \mathcal{Z}^{[m]\setminus v}_{\mathrm{dl}_K(v)} \to \mathcal{Z}^{[m]}_K$$

induces the same map as $\bar{\epsilon}_K$ in homology with $\Bbbk$ coefficient on the wedge summand $\bigvee_{I\subsetneq[m]} |\Sigma K_I| \wedge \widehat{X}^I$ of $\mathcal{W}^{[m]}_K$, where the last arrow is the wedge of inclusions. We here notice that there are many choices for the first arrow but any choice will do. Now our remaining task is to construct a map $|\Sigma K| \wedge \widehat{X}^{[m]} \to \mathcal{Z}^{[m]}_K$ which induces the same map as the restriction of $\bar{\epsilon}_K$ in homology with $\Bbbk$ coefficient. Define a map θ_v as the composite

$$\begin{aligned} |\Sigma \mathrm{dl}_K(v)| \wedge \widehat{X}^{[m]} \xrightarrow{\mathrm{incl}} \mathcal{W}^{[m]\setminus v}_{\mathrm{dl}_K(v)} \vee (\mathcal{W}^{[m]\setminus v}_{\mathrm{dl}_K(v)} \wedge X_v) \xrightarrow{\delta^{-1}} \mathcal{W}^{[m]\setminus v}_{\mathrm{dl}_K(v)} \rtimes X_v \\ \xrightarrow{\epsilon_{\mathrm{dl}_K(v)}\rtimes 1} \mathcal{Z}^{[m]\setminus v}_{\mathrm{dl}_K(v)} \rtimes X_v \xrightarrow{\mathrm{incl}} \mathcal{Z}^{[m]}_K/CX_m \xrightarrow{\simeq} \mathcal{Z}^{[m]}_K, \end{aligned}$$

where δ is as in (2.7) and the last arrow is the homotopy inverse of the projection $\mathcal{Z}^{[m]}_K \to \mathcal{Z}^{[m]}_K/CX_v$. By Proposition 2.4, we see that $\Sigma\theta_v$ is homotopic to

$$\Sigma|\Sigma \mathrm{dl}_K(v)| \wedge \widehat{X}^{[m]} \xrightarrow{\mathrm{incl}} \Sigma|\Sigma K| \wedge \widehat{X}^{[m]} \xrightarrow{\mathrm{incl}} \Sigma\mathcal{W}^{[m]}_K \xrightarrow{\bar{\epsilon}_K} \Sigma\mathcal{Z}^{[m]}_K.$$

Thus the composite

$$|\Sigma K| \wedge \widehat{X}^{[m]} \xrightarrow{s\wedge 1} \bigvee_{v\in[m]} |\Sigma \mathrm{dl}_K(v)| \wedge \widehat{X}^{[m]} \xrightarrow{\bigvee_{v\in[m]}\theta_v} \mathcal{Z}_K^{[m]}$$

is the desired map, and therefore the proof is completed. □

Corollary 3.3. *If K is an extractible complex on the index set $[m]$ and each X_i is a connected CW-complex, there is a homotopy equivalence*

$$\mathcal{Z}_K^{[m]} \simeq \mathcal{W}_K^{[m]}.$$

Proof. Combine Proposition 2.5, Theorem 3.2 and the J.H.C. Whitehead theorem. □

4. Shellable complexes

In this section, we prove that the Alexander duals of shellable complexes are extractible, which implies a wedge decomposition of polyhedral products for the Alexander dual of shellable complexes.

First of all, let us recall the definition of shellable complexes.

Definition 4.1. A simplicial complex K is called *shellable* if its facets are ordered as $F_1,\dots,F_t$ in such a way that the subcomplex $(\bigcup_{i=1}^{k-1} F_i)\cap F_k$ is pure and $(\dim F_k - 1)$-dimensional for $k=2,\dots,t$. Such an order $F_1,\dots,F_t$ is called a *shelling* of K.

We prepare two simple lemmas.

Lemma 4.2. *If a simplicial complex K is collapsible, $|K^\vee|$ is contractible.*

Proof. If $\sigma\in K$ is a free face of K and $\tau\in K$ satisfies $\sigma\subset\tau$ and $\dim\tau=\dim\sigma+1$, then it is straightforward to see that $\tau^\vee=[m]\setminus\tau$ is a free face of $(K\setminus\{\sigma,\tau\})^\vee$ and $\sigma^\vee=[m]\setminus\sigma\in(K\setminus\{\sigma,\tau\})^\vee$ satisfies $\tau^\vee\subset\sigma^\vee$ and $\dim\sigma^\vee=\dim\tau^\vee+1$. Then since $(K\setminus\{\sigma,\tau\})^\vee=K^\vee\cup\{\sigma^\vee,\tau^\vee\}$, if K is collapsible, $|K^\vee|$ has the homotopy type of a simplex which is contractible. □

Lemma 4.3. *Let K be a simplicial complex on the index set $[m]$ and choose the index set of $\mathrm{lk}_K(v)$ to be $[m]\setminus v$. Then*

$$\mathrm{lk}_K(v)^\vee = \mathrm{dl}_{K^\vee}(v.)$$

Proof. By definition, we have

$$\mathrm{lk}_K(v)^\vee = \{\sigma\subset[m]\setminus v \,|\, (([m]\setminus v)\setminus\sigma)\cup v\notin K\} = \{\sigma\subset[m]\setminus v\,|\,[m]\setminus\sigma\notin K\} = \mathrm{dl}_{K^\vee}(v).$$

□

We now prove the main result of this section. Given a shelling $F_1,\dots,F_t$ of K, F_k is called a spanning facet if the boundary of F_k is contained in $\bigcup_{i=1}^{k-1} F_i$. It is easy to see that if $F_{i_1},\dots,F_{i_k}$ are all spanning facets, $K\setminus\{F_{i_1},\dots,F_{i_k}\}$ is collapsible.

Proposition 4.4. *If K has no ghost vertex and $K^\vee$ is shellable, K is extractible.*

Proof. The proof is done by induction on m, where we put the index set of K to be $[m]$. The case $m = 1$ is trivial. Assuming the case $m-1$, we prove the case m. By Lemma 4.3, we have $\mathrm{dl}_K(v) = \mathrm{lk}_{K^\vee}(v)^\vee$. In [BW], it is shown that $\mathrm{lk}_{K^\vee}(v)$ is shellable, so $\mathrm{dl}_K(v)$ is extractible by the induction hypothesis. Let $F_1, \ldots, F_t$ be a shelling of $K^\vee$ and let $\Gamma_{K^\vee}$ be the set of all spanning facets of $K^\vee$ with respect to this shelling. Put $\Delta_{K^\vee} = K^\vee \setminus \bigcup_{F \in \Gamma_{K^\vee}} F$. Since each $F \in \Gamma_{K^\vee}$ is a minimal non-face of $\Delta_{K^\vee}$, we have

$$(\Delta_{K^\vee})^\vee = K^\vee \cup \bigcup_{F \in \Gamma_{K^\vee}} F^\vee, \tag{4.1}$$

where $F^\vee = [m] \setminus F$. Since $\Delta_{K^\vee}$ is collapsible, $|(\Delta_{K^\vee})^\vee|$ is contractible by Lemma 4.2, and then

$$|\Sigma K| \simeq |(\Delta_{K^\vee})^\vee|/|K| = \bigvee_{F \in \Gamma_{K^\vee}} S^{|F^\vee|} = \bigvee_{F \in \Gamma_{K^\vee}} S^{m-|F|}.$$

Let $F_{j_1}, \ldots, F_{j_l}$ be all facets of $K^\vee$ such that $v \in F_{j_i}$ and $j_1 < \cdots < j_l$. Then $F_{j_1} \setminus v, \ldots, F_{j_l} \setminus v$ is a shelling of $\mathrm{lk}_{K^\vee}(v)$, and with this shelling, $\Gamma_{\mathrm{lk}_{K^\vee}(v)} = \{F \setminus v \,|\, F \in \Gamma_{K^\vee},\ v \in F\}$. Choose the index set of $\mathrm{lk}_{K^\vee}(v)$ to be $[m] \setminus v$ and identify $\mathrm{lk}_{K^\vee}(v)^\vee$ with $\mathrm{dl}_K(v)$ by Lemma 4.3. Then by (4.1), we have the inclusion $\Delta_{\mathrm{lk}_{K^\vee}(v)} \to \Delta_{K^\vee}$ whose Alexander dual $(\Delta_{\mathrm{lk}_{K^\vee}(v)})^\vee \to (\Delta_{K^\vee})^\vee$ is the union of the inclusion $\iota : \mathrm{dl}_K(v) \to K$ and the identity map $([m] \setminus v) \setminus (F \setminus v) \to [m] \setminus F$ for $F \in \Gamma_{K^\vee}$ with $v \in F$. Then it induces a map

$$|(\Delta_{\mathrm{lk}_{K^\vee}(v)})^\vee|/|\mathrm{dl}_K(v)| \to |(\Delta_{K^\vee})^\vee|/|K| \tag{4.2}$$

which is identified with the inclusion

$$\bigvee_{\substack{F \in \Gamma_{K^\vee} \\ v \in F}} S^{m-|F|} \to \bigvee_{F \in \Gamma_{K^\vee}} S^{m-|F|}. \tag{4.3}$$

Consider the homotopy commutative diagram of homotopy cofiber sequences

$$\begin{tikzcd}
|\mathrm{dl}_K(v)| \arrow[r] \arrow[d, "|\iota|"] & |(\Delta_{\mathrm{lk}_{K^\vee}(v)})^\vee| \arrow[r] \arrow[d] & |(\Delta_{\mathrm{lk}_{K^\vee}(v)})^\vee|/|\mathrm{dl}_K(v)| \arrow[r] \arrow[d] & |\Sigma \mathrm{dl}_K(v)| \arrow[d, "|\Sigma\iota|"] \\
|K| \arrow[r] & |(\Delta_{K^\vee})^\vee| \arrow[r] & |(\Delta_{K^\vee})^\vee|/|K| \arrow[r] & |\Sigma K|.
\end{tikzcd}$$

Since $|(\Delta_{\mathrm{lk}_{K^\vee}(v)})^\vee|$ and $|(\Delta_{K^\vee})^\vee|$ are contractible, the right horizontal arrows are homotopy equivalences, so $|\Sigma\iota|$ is identified with the inclusion (4.2) hence with(4.3). Now it is easy to construct the desired map s and therefore the proof is completed. ☐

Proof of Theorem 1.2. Combine Corollary 3.3 and Proposition 4.4. ☐

Remark 4.5. In the proof of Theorem 4.4, we have actually proved that $|\Sigma K_I|$ has the homotopy type of a wedge of spheres for any $I \subset [m]$. Then Corollary 1.4 for the Alexander duals of shellable complexes follows from Theorem 1.2.

5. Sequentially Cohen-Macaulay complexes

In this section, we prove that the Alexander dual of sequentially Cohen-Macaulay complexes are extractible over $\mathbb{Z}_{(p)}$ for any prime p, which implies the p-local version of Theorem 1.2 for the Alexander dual of sequentially Cohen-Macaulay complexes. From this, we deduce Theorem 1.3 and Corollary 1.4 by applying the result of McGibbon [M] on the localization genus (or the Mislin genus).

Let us recall from [BWW] the definition of sequentially Cohen-Macaulay complexes. A space X is called n-acyclic over $\Bbbk$ if $\widetilde{H}_i(X;\Bbbk) = 0$ for $i \le n$. If $n = \infty$, we simply say X is acyclic over $\Bbbk$. For a simplicial complex K, let $K^{\langle i\rangle}$ denote the subcomplex of K generated by facets of dimension $\ge i$.

Definition 5.1. Let K be a simplicial complex.

1. K is called *sequentially acyclic* over $\Bbbk$ if $K^{\langle i\rangle}$ is $(i-1)$-acyclic over $\Bbbk$ for any i.
2. K is called *sequentially Cohen-Macaulay* over $\Bbbk$ if $\mathrm{lk}_K(\sigma)$ is sequentially acyclic over $\Bbbk$ for any simplex $\sigma \in K$, where σ can be $\emptyset$, or equivalently K itself is sequentially acyclic over $\Bbbk$.

As well as extractible complexes, if K is sequential Cohen-Macaulay over $\mathbb{Z}$, we simply say that K is sequentially Cohen-Macaulay. By definition, if K is sequentially Cohen-Macaulay over $\Bbbk$, so is $\mathrm{lk}_K(v)$ for any vertex v. We record an immediate consequence from this together with Lemma 4.3.

Lemma 5.2. *Let K be a simplicial complex on the index set $[m]$ and choose the index set of $\mathrm{dl}_K(v)$ to be $[m] \setminus v$. If $K^\vee$ is sequentially Cohen-Macaulay over $\Bbbk$, so is $\mathrm{dl}_K(v)^\vee$.*

The following simple lemma will be useful.

Lemma 5.3. *Let K be a simplicial complex with $\widetilde{H}_i(K^{\langle i+1\rangle};\Bbbk) = 0$. Then any i-cycle over $\Bbbk$ which is not a boundary involves a facet of dimension i.*

Proof. Let x be an i-cycle over $\Bbbk$. If x involves no facet, it is a cycle of $K^{\langle i+1\rangle}$ over $\Bbbk$. Then since $\widetilde{H}_i(K^{\langle i+1\rangle};\Bbbk) = 0$, x is a boundary, completing the proof. □

This lemma readily implies that if K is sequential Cohen-Macaulay over $\Bbbk$, $\widetilde{H}_*(K;\Bbbk)$ is a free $\Bbbk$-module. Let us further observe the homology of sequential Cohen-Macaulay complexes over $\Bbbk$. For a chain $x = \sum a_j\sigma_j$ of a simplicial complex K ($a_j \in \Bbbk$, $\sigma_j \in K$) and a vertex v, let $x_v = \sum_{v\in\sigma_j} a_j\sigma_j$.

Proposition 5.4. *If a cycle x of a simplicial complex K involves a facet F with $v \in F$, ∂x_v is a cycle of $\mathrm{lk}_K(v)$ which involves a facet $F \setminus v$ of $\mathrm{lk}_K(v)$ and is not a boundary.*

Proof. Consider the Mayer-Vietoris exact sequence

$$\cdots \to H_*(\mathrm{lk}_K(v);\Bbbk) \to H_*(\mathrm{dl}_K(v);\Bbbk) \bigoplus H_*(\mathrm{st}_K(v);\Bbbk) \to H_*(K;\Bbbk) \xrightarrow{\partial_*} H_{*-1}(\mathrm{lk}_K(v);\Bbbk) \to \cdots.$$

By a straightforward calculation, $\partial_*[x]$ is represented by ∂x_v. Notice that if a cycle involves a facet, it is not a boundary. By definition, ∂x_v involves a facet $F \setminus v$, therefore it is not a boundary, completing the proof. $\square$

We now want to prove that the Alexander duals of sequentially Cohen-Macaulay complexes over $\mathbb{Z}_{(p)}$ are extractible over $\mathbb{Z}_{(p)}$, but sequentially Cohen-Macaulay complexes are characterized by homology, not by facets, so we do not have the notion of spanning facets for sequentially Cohen-Macaulay complexes in general, which played the central role in the proof of Proposition 4.4. We then generalize the notion of spanning facets in a homological setting. Let K be a simplicial complex. Facets $F_1, \ldots, F_r$ of a simplicial complex K are called *spanning facets* over $\Bbbk$ if

(1) $\Delta_K = K \setminus \{F_1, \ldots, F_r\}$ is acyclic over $\Bbbk$.
(2) The boundary of F_i is in Δ_K for all i.

Let us recall the splitting of a suspension. For a space X, we choose one vertex from each path-component of X in such a way that the basepoint of X is involved. Let V be the set of these points. Then the homotopy cofiber sequence $\Sigma V \to \Sigma X \to \Sigma(X/V)$ splits as

$$\Sigma X \simeq \Sigma V \vee \Sigma(X/V),$$

which is natural with respect to X. Using this splitting, the (almost) p-localization of ΣX is defined as

$$\Sigma X_{(p)} = \Sigma V \vee \Sigma(X/V)_{(p)}.$$

Proposition 5.5. *If K has no ghost vertex and $K^\vee$ is sequentially Cohen-Macaulay over $\mathbb{Z}_{(p)}$, K is extractible over $\mathbb{Z}_{(p)}$.*

Proof. As is noted above, $H_*(K^\vee; \mathbb{Z}_{(p)})$ is a free $\mathbb{Z}_{(p)}$-module. Choose a basis $x_1^i, \ldots, x_{n_i}^i$ of $H_i(K^\vee; \mathbb{Z}_{(p)})$. By Lemma 5.3, x_1^i involves a facet F_1^i. Moreover, the modulo p reduction of $x_1^i, \ldots, x_{n_i}^i$ is a basis of $H_i(K^\vee; \mathbb{Z}/p)$, by normalizing x_1^i if necessary, we can choose F_1^i so that the coefficient of F_1^i is 1. Subtracting x_1^i from $x_2^i, \ldots, x_{n_i}^i$, we may assume that $x_2^i, \ldots, x_{n_i}^i$ do not involve F_1^i. Then by induction, we see that for $j = 1, \ldots, n_i$, x_j^i involves a facet F_j^i which is not involved in F_k^i for $k \neq j$ and the coefficient of F_j^i in x_j^i is 1.

Claim 5.6. *$F_1^1, \ldots, F_1^{n_1}, \ldots, F_1^d, \ldots, F_d^{n_d}$ are spanning facets of $K^\vee$ over $\mathbb{Z}_{(p)}$, where $d = \dim K$.*

Proof. The second condition for spanning facets is obviously satisfied. Put $\Gamma_{K^\vee}$ to be the set of all F_j^i and $\Delta_{K^\vee} = K^\vee \setminus \bigcup_{F \in \Gamma_{K^\vee}} F$. Then it remains to show that $\Delta_{K^\vee}$ is acyclic over $\mathbb{Z}_{(p)}$. Since each $F \in \Gamma_{K^\vee}$ satisfies the second condition for spanning facets, we have

$$|K^\vee|/|\Delta_{K^\vee}| = \bigvee_{F \in \Gamma_{K^\vee}} S^{|F|}$$

and by the choice of x_j^i, the pinch map $|K^\vee| \to |K^\vee|/|\Delta_{K^\vee}|$ sends x_j^i to a generator of $H_i(S^i; \mathbb{Z}_{(p)})$ in homology. Then the Puppe exact sequence

$$\cdots \to H_*(|\Delta_{K^\vee}|; \mathbb{Z}_{(p)}) \to H_*(|K^\vee|; \mathbb{Z}_{(p)}) \to H_*(|K^\vee|/|\Delta_{K^\vee}|; \mathbb{Z}_{(p)}) \to \cdots$$

shows that $\Delta_{K^\vee}$ is acyclic over $\mathbb{Z}_{(p)}$. $\square$

As in the proof of Proposition 4.4, we have

$$(\Delta_{K^\vee})^\vee = K \cup \bigcup_{F\in\Gamma_{K^\vee}} F^\vee, \quad \text{hence} \quad |(\Delta_{K^\vee})^\vee|/|K| = \bigvee_{F\in\Gamma_{K^\vee}} S^{m-|F|},$$

where $F^\vee = [m]\setminus F$. Consider the edges $F^\vee = [m]\setminus F$ of $(\Delta_{K^\vee}^\vee)$ for $F \in \Gamma_{K^\vee}$ with $|F| = m-2$. If two such edges have end points in the same component of K, we connect these ends by an edge path in K having no self-intersection and take one vertex from this path, where the connecting path is a single point if two ends coincide. Let V_K be the set of all these vertices and remaining ends of such edges. Then there is a one-to-one correspondence between V_K and the components of K, and we have constructed a tree T_K in $|(\Delta_{K^\vee})^\vee|$ whose vertex set is V_K. There is a homotopy commutative diagram of homotopy cofiber sequences

$$\begin{array}{ccccccc}
|V_K| & \longrightarrow & T_K & \longrightarrow & T_K/|V_K| & \xrightarrow{\tilde\alpha} & |\Sigma V_K| \\
\downarrow{\scriptstyle\iota} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle|\Sigma\iota|} \\
|K| & \longrightarrow & |(\Delta_{K^\vee})^\vee| & \longrightarrow & |(\Delta_{K^\vee})^\vee|/|K| & \xrightarrow{\alpha} & |\Sigma K| \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
|K|/|V_K| & \longrightarrow & |(\Delta_{K^\vee})^\vee|/T_K & \longrightarrow & |(\Delta_{K^\vee})^\vee|/(|K|\cup T_K) & \xrightarrow{\bar\alpha} & \Sigma(|K|/|V_K|).
\end{array} \tag{5.1}$$

Since T_K is contractible, $\tilde\alpha$ is a homotopy equivalence. Since $|(\Delta_{K^\vee})^\vee|$ is acyclic over $\mathbb{Z}_{(p)}$, α and then $\bar\alpha$ induces an isomorphism in homology with $\mathbb{Z}_{(p)}$ coefficient. Then since $|(\Delta_{K^\vee})^\vee|/(|K|\cup T_K)$ is simply connected, we get a homotopy equivalence

$$|\Sigma K|_{(p)} \simeq \bigvee_{\substack{F\in\Gamma_{K^\vee}\\ |F|=m-2}} S^{m-|F|} \vee \bigvee_{\substack{F\in\Gamma_{K^\vee}\\ |F|<m-2}} S^{m-|F|}_{(p)}. \tag{5.2}$$

We induct on m to prove Proposition 5.5. The case $m = 1$ is trivial. Suppose the case $m-1$ holds. By Lemma 5.2 and the induction hypothesis, $\mathrm{dl}_K(v)$ is extractible for any vertex v. By Lemma 5.4, if $F_1,\ldots,F_r \in \Gamma_{K^\vee}$ involve a vertex v, $\partial(x_1)_v,\ldots,\partial(x_r)_v$ form a part of a basis of $H_*(\mathrm{lk}_{K^\vee}(v);\mathbb{Z}_{(p)})$, where x_i is a cycle corresponding to F_i. Then in the above way, we can choose spanning facets of $\mathrm{lk}_{K^\vee}(v)$ which include $F\setminus v$ for all $F\in\Gamma_{K^\vee}$ with $v\in F$. Hence with these spanning facets, we have the homotopy commutative diagram (5.1) for $\mathrm{lk}_{K^\vee}(v)^\vee = \mathrm{dl}_K(v)$ which is compatible with that for K, so through the homotopy equivalence (5.2), the inclusion $|\Sigma\mathrm{dl}_K(v)|_{(p)} \to |\Sigma K|_{(p)}$ is identified with the wedge of the identity map of $\bigvee_{\substack{F\in\Gamma_{K^\vee}\\ v\in F}} S^{m-|F|}$ and the constant map on the remaining wedge summand. It is now easy to construct the desired map, completing the proof. $\square$

The following theorem is an immediate consequence of Corollary 3.3 and Proposition 5.5.

Theorem 5.7. *Let K be a simplicial complex on the index set $[m]$. If $K^\vee$ is sequentially Cohen-Macaulay and each X_i is a connected CW-complex, then for any prime p, there is a*

p-local homotopy equivalence

$$\mathcal{Z}_K^{[m]} \simeq_{(p)} \mathcal{W}_K^{[m]}.$$

We want to integrate the p-local homotopy equivalences of Theorem 5.7. To this end, let us recall the result of McGibbon [M] on the relation between localization and co-H-structures.

Proposition 5.8. *Let X, Y be simply connected finite complexes. If $X \simeq_{(p)} Y$ for any prime p and Y is a co-H-space, X is also a co-H-space.*

Proof of Theorem 1.3. By Theorem 5.7, $\mathcal{Z}_K^{[m]} \simeq_{(p)} \mathcal{W}_K^{[m]}$ for any prime p and $\mathcal{W}_K^{[m]}$ is a suspension. Since each X_i is a finite complex, so are $\mathcal{Z}_K^{[m]}$ and $\mathcal{W}_K^{[m]}$. Then by Proposition 5.8, $\mathcal{Z}_K^{[m]}$ is a co-H-space. Therefore Theorem 1.3 follows from Corollary 2.6. □

In order to prove Corollary 1.4, we prepare the following simple lemma.

Lemma 5.9. *Let X be a connected finite type CW-complex. If $\Sigma X_{(p)}$ has the homotopy type of a wedge of p-local spheres for any prime p, ΣX itself has the homotopy type of a wedge of spheres.*

Proof. By assumption, $H_i(\Sigma X; \mathbb{Z})$ is a free abelian group of finite rank. Choose a basis $x_1^i, \ldots, x_{n_i}^i$ of $H_i(X; \mathbb{Z})$. Using a p-local homotopy equivalence between ΣX and a wedge of spheres, we can easily construct a map ${}_p\theta_j^i : S^i \to \Sigma X_{(p)}$ satisfying $({}_p\theta_j^i)_*(u_i) = x_j^i$ in homology with $\mathbb{Z}_{(p)}$ coefficient for any i, j, where u_i is a generator of $H_i(S^i; \mathbb{Z}) \cong \mathbb{Z}$. Let $\{p_1, p_2, \ldots\}$ be the set of all primes with $p_i \neq p$. It is well known that $\Sigma X_{(p)}$ is given as the homotopy colimit of the sequence

$$\Sigma X \xrightarrow{\underline{l_1}} \Sigma X \xrightarrow{\underline{l_2}} \Sigma X \xrightarrow{\underline{l_3}} \Sigma X \xrightarrow{\underline{l_4}} \cdots$$

where $l_k = p_1 \cdots p_k$ and $\underline{q} : \Sigma X \to \Sigma X$ is the degree q map. By the compactness of S^i, ${}_p\theta_j^i$ factors through the finite step of the above sequence. Then there is a map ${}_p\bar{\theta}_j^i : S^i \to \Sigma X$ satisfying $({}_p\bar{\theta}_j^i)_*(u_i) = {}_pa_j^i x_j^i$ with $p \nmid {}_pa_j^i$ in the integral homology. Now we can choose primes $q_1, \ldots, q_n$ such that ${}_{q_1}a_j^i, \ldots, {}_{q_n}a_j^i$ are relatively prime. There are integers $d_1, \ldots, d_n$ such that $d_1({}_{q_1}a_j^i) + \ldots + d_n({}_{q_n}a_j^i) = 1$ hence the map

$$\lambda_j^i = \underline{d_1} \circ {}_{q_1}\bar{\theta}_j^i + \cdots + \underline{d_n} \circ {}_{q_n}\bar{\theta}_j^i,$$

satisfies $(\lambda_j^i)_*(u_i) = x_j^i$ in the integral homology, where the sum is defined by using the suspension comultiplication of ΣX. Thus the map $\bigvee_{i \geq 1} \bigvee_{j=1}^{n_i} \lambda_j^i$ induces an isomorphism in the integral homology, and therefore the proof is completed by the J.H.C. Whitehead theorem. □

Proof of Corollary 1.4. By Theorem 1.3, there is a homotopy equivalence

$$\mathcal{Z}_K(\underline{D}^2, \underline{S}^1) \simeq \bigvee_{\emptyset \neq I \subset [m]} |\Sigma K_I| \wedge S^{|I|},$$

where we exclude the case $I = \emptyset$ since the corresponding wedge summand is a single point. By Lemma 5.2 and (5.2), $\Sigma|\Sigma K_I|_{(p)}$ has the homotopy type of a wedge of p-local spheres for any

prime p and $I \subset [m]$. Then it follows from Lemma 5.9 that $\Sigma|\Sigma K_I|$ has the homotopy type of a wedge of spheres for any $I \subset [m]$ hence the result. $\square$

References

[BBCG] A. Bahri, M. Bendersky, F. R. Cohen, and S. Gitler, *The polyhedral product functor: a method of decomposition for moment-angle complexes, arrangements and related spaces*, Advances in Math. **225** (2010), 1634-1668.

[B] A. Berglund, *Shellability and the strong gcd-condition*, Elect. J. Comb. **16** (2009), 1-7.

[BW] A. Björner and M.L. Wachs, *Nonpure shellable complexes and posets II*, Trans. AMS **349** (1997), 3945-3975.

[BWW] A. Björner, M. Wachs, and V. Welker, *On sequentially Cohen-Macaulay complexes and posets*, Israel J. Math. **169** (2009), 295-316.

[BP] V.M. Buchstaber and T.E. Panov, *Torus actions and their applications in topology and combinatorics*,University Lecture Series **24**, American Mathematical Society, Providence, RI, 2002.

[DJ] M.W. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Mathematical Journal **62** (1991), 417-452.

[G] E.S. Golod, *On the homologies of certain local rings*, Soviet Math. Dokl. **3** (1962), 745-748.

[GT] J. Grbić and S. Theriault, *The homotopy type of the complement of a coordinate subspace arrangement*, Topology **46** (2007), 357-396.

[GW] V. Grujić and V. Welker, *Discrete Morse theory for moment-angle complexes of pairs* (D^n, S^{n-1}), arXiv:1212.2028.

[HRW] J. Herzog, V. Reiner, and V. Welker, *Componentwise linear ideals and Golod rings*, Mich. Math. J. **46** (1999), 211-223.

[H] M. Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, in Proceedings of the Second Oklahoma Ring Theory Conference (March 1976), Marcel-Dekker, New York, 1977, 171-223.

[IK] K. Iriye and D. Kishimoto, *Decompositions of polyhedral products for shifted complexes*, available at http://www.math.kyoto-u.ac.jp/preprint/preprint2011.html.

[M] C.A. McGibbon, *On the localization genus of a space*, Algebraic Topology: New Trends in Localization and Periodicity (Sant Feliu de Guídols, 1994), Birkhäuger, Basel, 1996, 285-306.

[P] T. Porter, *Higher-order Whitehead products*, Topology **3** (1965), 123-135.

[ZZ] G.M. Ziegler and R.T. Živaljević, *Homotopy types of subspace arrangements via diagram of spaces*, Math. Ann. **295** (1993), 527-548.

Department of Mathematics and Information Sciences, Osaka Prefecture University, Sakai, 599-8531, Japan

E-mail address: kiriye@mi.s.osakafu-u.ac.jp

Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan

E-mail address: kishi@math.kyoto-u.ac.jp